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EXPLICIT SOLUTIONS FOR SEEPAGE INFILTRATING INTO A POROUS EARTH DAM DUE TO PRECIPITATION

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SUMMARY

Steady two-dimensional gravity-driven seepage in homogeneous porous lumps is studied with the help of conformal mappings and boundary value problem technique. The Terzaghi flow pattern for a trapezoidal dam exposed to a heavy rainstorm is analysed. For a semi-circular massif, the influence of impervious bed inclination is studied. Recharge–discharge distributions, hinge points, gradients along the lump contour as well as the total flow rate exhibiting water-bearing capacity of the unit are found in explicit form. Generalizations for non-isobaric boundary conditions are discussed.

KEY WORDS: seepage; recharge–discharge; hydraulic gradient; conformal mapping; boundary value problems

INTRODUCTION

Gravity-driven seepage flows in permeable formations caused by precipitation–evaporation attract the attention of hydrogeologists and civil engineers. First, one considers regional flows under approximate or even uncertain input data on topography, recharge–discharge spatial and temporal distributions, permeability, porosity, etc. The aims of the analysis include the characterization of hydrodynamical patterns of subsurface flows, estimations of groundwater budgets for contaminant transport and heat transfer scenarios, etc.[1,2] Soil mechanics deals with earth dams, hillslopes, draining ditches, etc. and civil engineers have usually more information about both porous media and other factors influencing seepage. Their main goal is safety of the structure under design or observation, in particular, slope stability against hydraulic gradients and their possible sudden changes[3]. These changes are often due to rainstorms or speed drawdowns in reservoirs adjacent to a porous earthwork.[4] Investigations of mechanical damage due to seepage and chemical movement in permeable formations as a first step call for a description of hydraulic parameters (heads, stream lines, seepage velocities and pressures). Having established the flow pattern, one may either draw simple conclusions based only on the hydrodynamical (advective) picture[5] or solve the next problem, for example, study stability[6] or contaminant dispersion.[7] In what follows, a simple analytical method is presented to determine the hydraulic parameter patterns within homogeneous permeable lumps under assumption of the Darcy law. We refer only to the bibliography concerning analytical approaches, omitting the many papers with numerical analysis.

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Muskat[8] seems to be the first who presented an analytical solution for a flow in a porous hillside exposed to rainfall. His 1-D approach assumes the entire hill being fully saturated with recharge boundary condition along the slope. In his classical work Hubbert[9] has given 2-D flow nets in an infinitely deep and homogeneous layer with wavy topography and intermittently changing recharge–discharge zones. The mathematical model for this flow pattern was developed by Toth.[2] For finite size and multi-layered water-bearing units the seepage picture may be intriguingly difficult, exhibiting local–subregional–regional fluxes, dividing lines, hinge points, stagnation points.[2,5,10–13] In most studies of recharged–discharged units the analytical technique is based on the Gramm–Schmidt orthonormalization procedure. It assumes the boundary under precipitation being an isobaric line (seepage face with accretion) as in the Muskat, Toth and Hubbert works. That allows for the use of harmonic functions in prescribed zones neglecting phreatic surfaces and unsaturated zones which make the problem non-linear (even in the simplest schemes problems including recharge along free surfaces are very difficult for analytical treatment[14,15]). Warrick[16] has employed a different approach based on conformal mappings of the flow region and the Zhukovskii plane. It should be noted again that large-scale hydrogeological models for regional flows[1,2] are completely analogous to the small-scale models[4] of seepage within earth embankments.

In this paper, we utilize the method of Warrick to study 2-D steady seepage in fully saturated ground massifs. Namely, we combine a conformal mapping with solution of a mixed boundary-value problem as we did for free surface and confined flows.[17,18] This provides answers to the following questions: What is the total water-bearing capacity of a lump? In other words, how much water can pass through a saturated unit? What is the inflow–outflow distribution along the lump boundary? In particular, where are the hinge points located? How do shape of the lump and bed inclination angle influence the characteristics mentioned? First, we consider seepage through a trapezoidal embankment exposed to a heavy rainstorm that was qualitatively described by Terzhagi and Peck.[19] Then we discuss application to the case of nonzero water level in pools. Finally, we study a semi-circular lump on an impervious inclined bed.

SEEPAGE IN A TRAPEZOIDAL DAM

We consider a trapezoidal dam $ABCD$ on a horizontal impermeable bottom AB. Designate the dam slope angle as α, geometrical sizes as l and H (Figure 1(a)), hydraulic conductivity as k. During a heavy rainstorm water enters the hill through the crest and the slopes, and drains out through the lower part of the slopes.[4] The part $BCDA$ of the dam contour contacting with air is assumed to be an isobaric line. Along the dividing line, ON, the flow is bisected: one part seeps leftward, another one rightward. H_1 and H_2 are hinge points: along H_1CDH_2 water penetrates the dam, along BH_1 and AH_2 it seeps out. Designate as $2q$ the total quantity of water seeping through the dam (per unit length). Let us introduce the reduced (i.e. divided by k) complex potential $w(z) = \phi + i\psi$, where $z = x + iy$, $\phi = -h$, h is the hydraulic head, and ψ is the reduced stream function. First, we map the flow region $ABCD$ onto the half-plane $\operatorname{Im} \zeta > 0$ of the auxiliary plane $\zeta = \xi + i\nu$ (Figure 1(b)) by the Schwarz–Christoffel transformation:[20]

$$z = P\int_0^{\zeta} (c^2 - \tau^2)^{\alpha}(1 - \tau^2)^{1-\alpha}\,d\tau \tag{1}$$

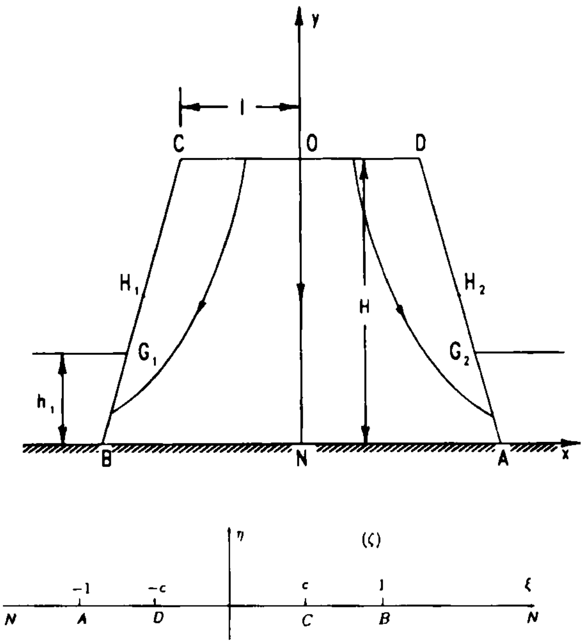

Figure 1. Flow domain (a) and an auxiliary plane (b) for seepage through a soil dam

where transform parameters are derived in the usual way as follows:

$$P = -\frac{1}{\int_0^c (c^2 - \tau^2)^\alpha (1 - \tau^2)^{1-\alpha}\,d\tau}$$

$$\frac{H}{l} = \sin(\alpha\pi)\frac{\int_c^1 (\tau^2 - c^2)^\alpha (1 - \tau^2)^{1-\alpha}\,d\tau}{\int_0^c (c^2 - \tau^2)^\alpha (1 - \tau^2)^{1-\alpha}\,d\tau} \tag{2}$$

Given the geometrical parameters (α, H, l) we derive c and P from (2). From (1) we obtain for $\zeta \to \xi$, $-1 \leqslant \xi \leqslant -c$ and $c \leqslant \xi \leqslant 1$ the expression for $y(\xi)$ along AD and CB, respectively (we omit them for brevity since we will actually need only $y'(\xi)$).

The next step of the solution is derivation of the second analytic function. For this purpose Warrick[16] has used the condition $p/\gamma = -(\phi + y) = 0$, γ is the specific weight of water, along the upper boundary of the flow region, mapped the Zhukovskii plane onto a half-plane, and eliminating ζ obtained an explicit formula for all the seepage characteristics. We shall use a different technique even though we may also construct the hodograph plane for this case[15] and derive the second conformal mapping. Namely, we solve a boundary value problem for $w(\zeta)$. Boundary conditions for this function are as follows:

$$\phi = 0 \text{ along } c < \xi < c, \qquad \phi = -y(\xi) \text{ along } c \leqslant |\xi| \leqslant 1, \qquad \psi = 0 \text{ along } |\xi| \geqslant 1 \tag{3}$$

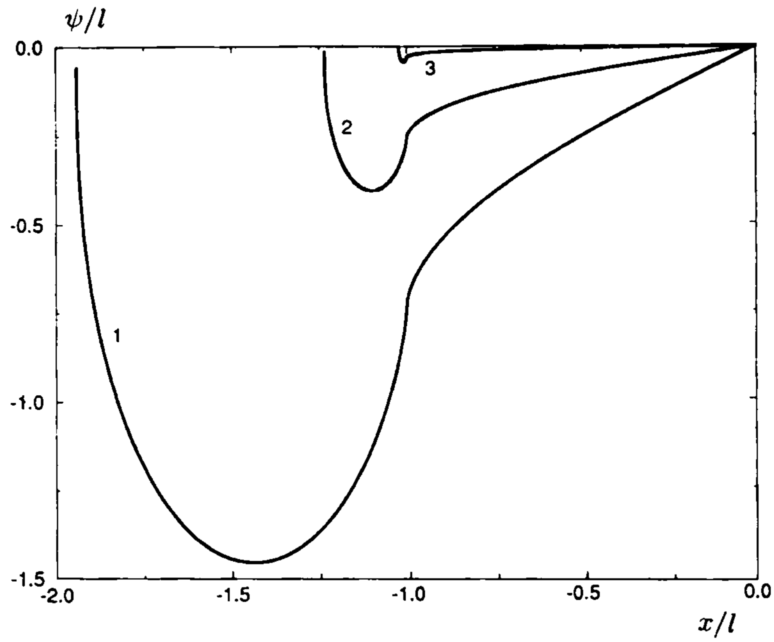

Figure 2. Recharge–discharge intensity ψ/l along the trapezoidal dam contour as a function of x/l for three values of H/l

Thus (3) describes a mixed boundary value problem whose solution is[15]

$$w = \frac{\sqrt{(\zeta^2 - 1)}}{\pi}\left[\int_{-1}^{-c} \frac{y(\tau)\,d\tau}{(1-\tau^2)^{1/2}(\tau-\zeta)} + \int_{c}^{1} \frac{y(\tau)\,d\tau}{(1-\tau^2)^{1/2}(\tau-\zeta)}\right] \tag{4}$$

From (1) and (4) specific discharge V and its horizontal u and vertical v components are easily determined as $\bar{V}(\zeta) = (dw/d\zeta)(dz/d\zeta)^{-1}$ where overbar means complex conjugation.

At $\zeta \to \xi$, $-1 \leqslant \xi \leqslant 1$ we obtain from (4) an equation for the stream function that differs from (4) only in substitutions $(\zeta^2 - 1)^{1/2} \to (1 - \xi^2)^{1/2}$, $\zeta \to \xi$. Integrating by parts this equation we derive

$$\psi = P\pi^{-1}\sin(\alpha\pi)\left[-\int_{-1}^{-c} F(\xi,\tau)\,d\tau + \int_{c}^{1} F(\xi,\tau)\,d\tau\right]$$

$$F(\xi,\tau) = y'(\tau)\ln\left|\frac{1 - \xi\tau + (1-\xi^2)^{1/2}(1-\tau^2)^{1/2}}{\tau - \xi}\right| \tag{5}$$

and $y'(\tau) = P\sin(\alpha\pi)(\tau^2 - c^2)^{\alpha}(1-\tau^2)^{1-\alpha}$ according to (1). Equation (5) provides the stream function distribution along $ABCD$. Note that during computations of the integrals we separated singularities at $\tau \to \xi$ in a routine manner. These singularities exist only for the segments BC and AD, along CD we have ordinary integrals.

Figure 2 represents $\psi(x)$ distributions along BCO with $x(\xi)$ being calculated again from (1) for the following parameters: $\alpha = 3\pi/8$ and $H/l = 2{\cdot}39, 0{\cdot}6, 0{\cdot}05$ (curves 1–3, respectively). Obviously, the minima on the graphs correspond to the hinge point H_1. From the figures plotted, we see that for small x recharge distribution is nearly linear, which represents the practically sound case of spatially uniform rainfall.

SEEPAGE IN A SEMI-CIRCULAR MASSIF

In the previous section we mapped the z-region on the ζ-plane and put the corresponding function into the kernel of integral representation of a mixed boundary value problem. We shall demonstrate now how conformal mapping may be replaced by the solution of another boundary value problem. It makes possible the study of a broad class of arbitrary curvilinear massif boundaries. However, for simplicity we consider a semi-circular massif BA with radius r placed on an impervious bed BNA with inclination angle β to the horizon Nx^* (Figure 3) such that the circle centre coincides with the origin of the two co-ordinate systems, xNy and x^*Ny^*, and polar angle θ is measured from the Nx line. Obviously, $y^* = y\cos(\beta) + x\sin(\beta)$. Bed inclination makes the dividing line D_1D_2 curvilinear and the hinge points H_1 and H_2 are spaced non-symmetrically on the outline of the massif.

In the auxiliary ζ-plane we assume, for the isobaric segment, BH_1H_2A of an arbitrary curvilinear massif $y = y(\xi)$, $-1 \leqslant \xi \leqslant 1$. Then for $z(\zeta)$ we have a Dirichlet boundary value problem: $y = 0$ at $|\xi| \leqslant 1$ and $y = y(\xi)$ at $|\xi| \leqslant 1$. Therefore, the integral solution is[15]

$$z = \frac{1}{\pi}\int_{-1}^{1} \frac{y(\tau)}{\tau - \zeta}\,\mathrm{d}\tau \tag{6}$$

Let us use the Chebyshev series expansion for the kernel of the integral in (6). Namely, express $y = \sum b_n U_n(\xi)$ with summation here and below from $n = 1$ to ∞. Then, the real part of z is $x = -\sum b_n T_n(\xi)$ where $U_n = \sin(n\arccos(\xi))$, $T_n = \cos(n\arccos(\xi))$. Choosing values of b_n we may construct various lump shapes including hummocky profiles. However, as we have said, we confine ourselves with the simplest case putting $b_1 = r$, and $b_n = 0$ for $n = 2, 3, \ldots$ In this case BA is a semi-circle:

$$y = r\sqrt{(1 - \xi^2)}, \quad x = -r\xi \tag{7}$$

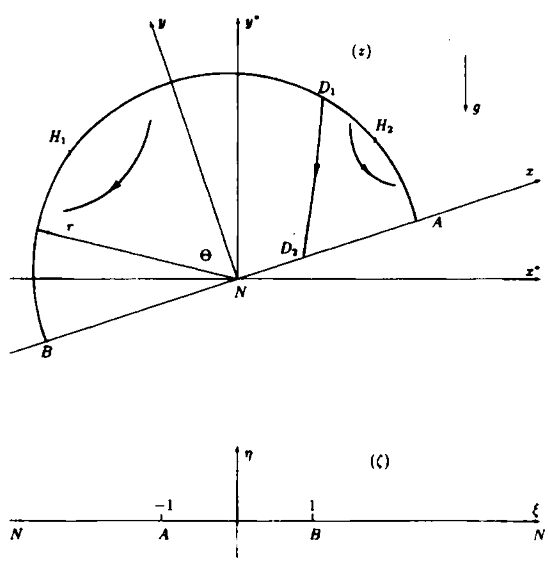

Figure 3. Flow domain and an auxiliary plane for seepage through a semi-circular massif

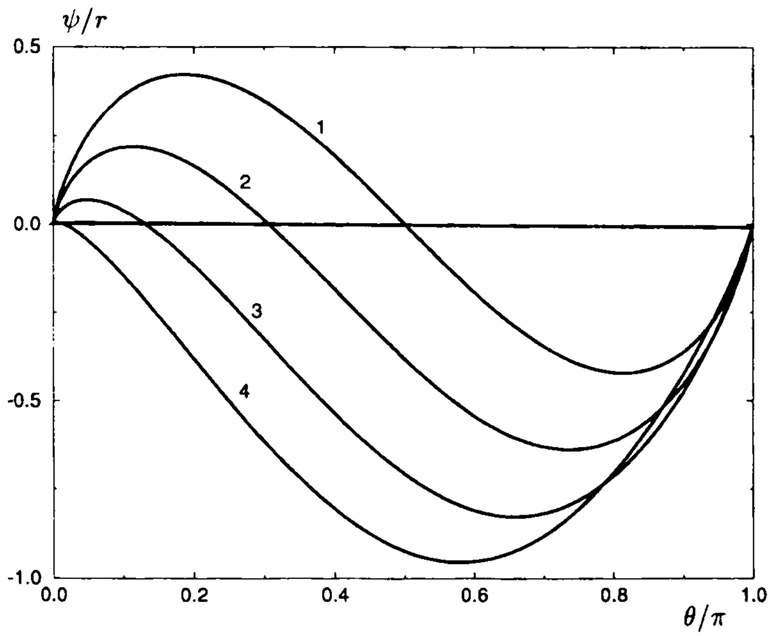

Figure 4. Recharge discharge intensity ψ/r along the circular massif contour as a function of polar angle θ/π for four values of the bed inclination angle β

For $w(\xi)$ we have a mixed boundary value problem with the following boundary conditions: $\psi = 0$ at $|\xi| \geqslant 1$, $\phi = -y(\xi)\cos(\beta) - x(\xi)\sin(\beta)$ at $|\xi| \leqslant 1$ where $x(\xi)$ and $y(\xi)$ are taken from (7). The solution to this problem is[15]

$$w = \frac{\sqrt{(\zeta^2 - 1)}}{\pi} \int_{-1}^{1} \frac{\phi(\tau)\,\mathrm{d}\tau}{\sqrt{(1 - \tau^2)}(\tau - \zeta)} \tag{8}$$

Substituting (7) into (8) and passing to the limit $\zeta \to \xi$, $-1 \leqslant \xi \leqslant 1$ we obtain for recharge–discharge distribution along the contour of our semi-circle

$$\psi = r\cos(\beta)\frac{\sqrt{(1 - \xi^2)}}{\pi}\ln\frac{1 - \xi}{1 + \xi} - r\sin(\beta)\sqrt{(1 - \xi^2)} \tag{9}$$

Point D_1 on the dividing line that bisects the flow into downward and upward seeping parts is found from (9) as solution of the equation $\psi = 0$. Differentiating (9) we derive the hinge points H_1 and H_2 as the roots ξ_1, ξ_2 of the equation $\psi'(\xi) = 0$. For the case $\beta = 0$, this equation is especially simple, $\tan(\theta/2) = \exp(-0{\cdot}5/\cos(\theta))$, $\theta = \text{across}(-\xi)$, and its two roots are $\theta = 33.5$ and 146.4° The total flow rate through our massif is $Q = \psi(\xi_2) - \psi(\xi_1)$. Figure 4 illustrates the distribution of recharge–discharge intensity ψ/r as functions of θ/π for $\beta = 0$, $\pi/8$, $\pi/4$, $3\pi/8$ (curves 1–4, respectively). Clearly, with increase of inclination angle we go from a symmetrical flow to flow patterns with low or no inflow through the segment near the point A and intensive outflow through the downstream part of the lump. The normal component J_n of the flow gradient J along AFB is found as $J_n = \mathrm{d}\psi/\mathrm{d}n = r^{-1}\,\mathrm{d}\psi/\mathrm{d}\theta$. Figure 5 shows J_n as a function of θ/π for the same values of β as in Figure 4. In these graphs $J_n = 0$ corresponds to the hinge points.

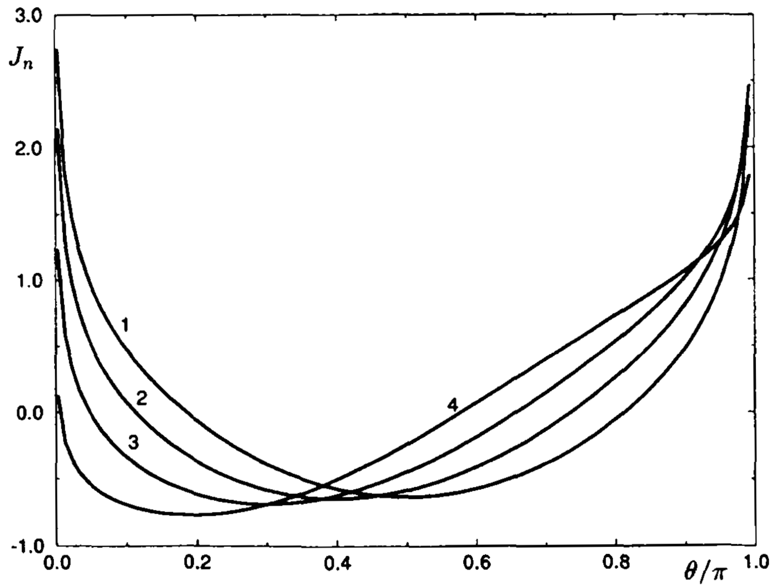

Figure 5. Normal gradient J_n along the massif contour as a function of θ/π

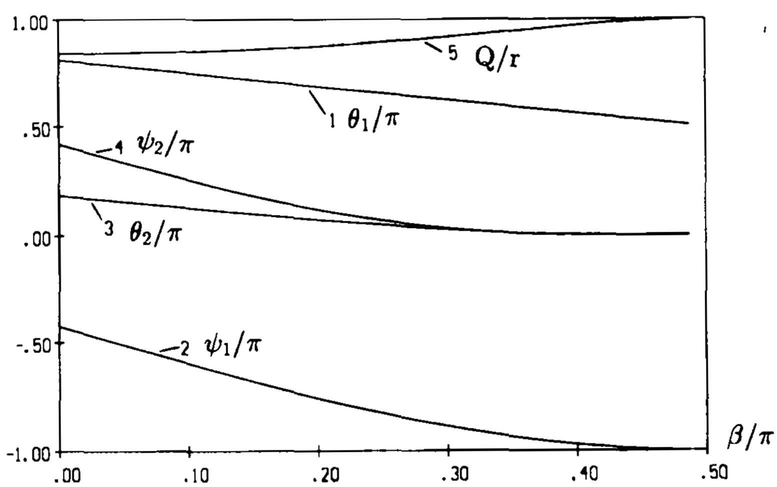

Figure 6. Angular co-ordinate θ_1 and θ_2 of the hinge points (curves 1 and 3), the corresponding stream function values ψ_1 and ψ_2 (curves 2 and 4), and the total flow rate Q/r through the massif as functions of β/π

For stability analysis of most interest are the segments BH_1 and AH_2 where water seeps out. The graphs depicted show high values of J_n along these areas that is in agreement with the known precaution against heavy rainstorms or reservoir water level oscillations for unprotected soil embankments.[4] Figure 6 illustrates dependencies of the angular co-ordinates θ_1/π and θ_2/π of the hinge points H_1 and H_2 (curves 1 and 3), the corresponding values of stream function ψ_1/r and ψ_2/r (curves 2 and 4), and Q/r (curves 5) on β. Obviously, at $\beta = \pi/2$ we come to a 1-D flow.

DISCUSSION AND CONCLUSIONS

The technique described above is easily applied to many other lump geometries. As far as conformal mapping is constructed we need only calculation of Cauchy-type integrals.

Another generalization of the method implemented involves the consideration of constant head boundaries. For example, if water levels in dam pools are h_1 (Figure 1(a)) we have a condition $\phi = h_1 - H$ along the corresponding segments BG_1 and AG_2 and a common seepage face condition along G_1CDG_2. It does not bring any difficulties except that we have to solve a system of two non-linear equations to derive the two conformal mapping parameters (we omit the corresponding algebra).

Recharge–discharge distribution both for the trapezoidal dam and the semi-circular lump involves an intrinsic question about their reality. So Kezdi and Rethati[4] state: 'In order to maintain a permanent flow, a certain intensity of rain is necessary.' Similarly, Angeli[21] indicates that boundary conditions for this type of problem should be taken from observations (though it is hardly available in detail practically). Natural recharge intensity will certainly differ from pictures in Figures 2 and 4 or analogous spatial distributions derived by Selim[12] and others. In reality $\psi(x)$ results from hydrological data and one of its approximations is a stepwise-constant function.[15] In this case we have surely to include free surfaces into analysis. Even then the analytical approach (for example, the hodograph method) works well under the additional assumption. For instance, in the scheme considered by Foster and Smith[14] recharge distribution between a hinge point and the point of detachment of a free surface is non-uniform, and if derived analytically leads to a distribution which follows from the solution. Epitomizing, *a posteriori* calculated recharge distribution is 'a payment' for ignorance of the unsaturated flow and free surfaces.

A fully saturated regime assumed is in some sense an ultimate one. Namely, it seems reasonable that increasing recharge above the level depicted in Figures 2 and 4, we will come to a runoff scenario when a lump under study cannot bear the whole quantity of water precipitated. Symmetrically, a decrease in this intensity will probably produce free surfaces. However, rigorous analysis is needed for these estimations.

Stability analysis includes many phenomena (sliding, piping, suffusion, heaving, etc.[4]) caused by or interconnected with seepage. For this purpose it is necessary to have the values of pressures and seepage velocities within the whole domain. Unfortunately, for arbitrary lumps extension of the method presented to the analysis of 'inner' porous areas seems to be somewhat onerous (recall that we derived hydraulic parameters along the boundary of our seepage region). However, even for stability assessments it may be of interest to reproduce distribution of gradients along the porous region profile.[22]

Real porous lumps are non-homogeneous and have more complicated topology than those that have been considered above. However, the explicit fast analytical solutions considered may be of interest as check procedures during the implementation of numerical calculations for practically significant situations.

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